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On a class of bidimensional nonseparable wavelet multipliers ☆

Yunzhang Li

Department of Applied Mathematics, Beijing Polytechnic University, Beijing, 100022, PR China

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Abstract

Let

$$E = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

A measurable function v is called an *E-wavelet multiplier* if $(v\hat{\psi})\check{}$ is an E-wavelet whenever ψ is an E-wavelet. In this paper, some characterizations of E-wavelet multiplier are obtained. As an application of these techniques, we prove that the set of E-wavelets is arcwise connected. © 2002 Elsevier Science (USA). All rights reserved.

Keywords: Wavelet multiplier; Multiresolution analysis

1. Introduction

Let

$$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{1.1}$$

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E-mail address: yzlee@bjpu.edu.cn.

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and

$$D = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}. \tag{1.2}$$

Throughout this paper, we restrict ourselves to the case $E = M$ or D.

A ladder of closed subspaces $\{V_j\}_{j\in Z}$ of $L^2(R^2)$ is called a *multiresolution analysis* related to E (MRA) if the following conditions hold:

(1) $V_j \subseteq V_{j+1}$ for $j \in Z$;
(2) $\bigcap_{j\in Z} V_j = \{0\}$, $\overline{\bigcup_{j\in Z} V_j} = L^2(R^2)$;
(3) $f(\cdot) \in V_j$ if and only if $f(E^{-j}\cdot) \in V_0$ for $j \in Z$;
(4) there exists a function $\phi(\cdot)$ in V_0 such that the set $\{\phi(\cdot - k)\}_{k\in Z^n}$ is an orthonormal basis for V_0.

Here $\phi(\cdot)$ is also called an *E-scaling function* of the MRA. Since $V_0 \subseteq V_1$, $\phi(\cdot)$ has to satisfy some *E-refinement equation*

$$\phi(\cdot) = |\det E| \sum_{n\in Z^2} h_n \phi(E \cdot - n). \tag{1.3}$$

We call

$$H_0(\cdot) = \sum_{n\in Z^2} h_n e^{-in\cdot} \tag{1.4}$$

a low pass *E-filter*.

An MRA related to E can derive an wavelet [1], which is called an *MRA E-wavelet*. Moreover, the wavelet has explicit expression then.

Before proceeding further, we introduce some notations and notions. For any two Lebesgue measurable functions f and g, $f = g$ means that $f(\cdot) = g(\cdot)$ up to a set of Lebesgue measure zero. For $S_1, S_2 \subseteq R^2$, $S_1 = S_2$ means that $S_1 = S_2$ up to a set of Lebesgue measure zero. For any 2×2 matrix A, A^* denotes the transpose of A.

A collection of sets $\{S_j\colon j \in J\}$ is called a *partition* of R^2, if $S_j \cap S_k = \emptyset$ for $j \neq k$, and $\bigcup_{j\in J} S_j = R^2$.

For $f \in L^1(R^2)$, we define the *Fourier transform* of f by

$$\hat{f}(\xi) = \int_{R^2} dx\, f(x) e^{-ix\xi}.$$

A measurable function ψ is called an *E-wavelet* if $\{\psi_{j,k}(\cdot)\colon j \in Z,\ k \in Z^2\}$ is an orthonormal basis in $L^2(R^2)$, where $f_{j,k}(\cdot) = 2^{j/2} f(E^j \cdot - k)$ for $j \in Z$, $k \in Z^2$, and any function f defined on R^2. It is well known that an E-wavelet is not necessarily derived from an MRA related to E.

A measurable function v is called an *E-wavelet multiplier* (*MRA E-wavelet multiplier*) if $(v\hat{\psi})\check{}$ is an E-wavelet (MRA E-wavelet) whenever ψ is an E-wavelet (MRA E-wavelet). v is called an *E-scaling function multiplier* if $(v\hat{\phi})\check{}$ is an E-scaling function whenever ϕ is an E-scaling function. A measurable function μ is called a *low pass E-filter multiplier* if μH_0 is a low pass E-filter whenever H_0 is a low pass E-filter.

[2] is the first of a series of reports describing joint results by two groups, one led by Xingde Dai and David Larson, and the other led by Eugenio Hernández and Guido Weiss. In that paper, some characterizations of one-dimensional 2-band wavelet multipliers were obtained, and, as an application, the arcwise connectivity of the set of one-dimensional 2-band MRA wavelets were also obtained. Then, as a mathematical problem, it is natural to address their counterparts in other cases. In present paper, some characterizations of E-multipliers are obtained. What is more, we prove that the set of E-wavelets is arcwise connected in $L^2(R^2)$.

We deliberately restrict ourselves to the dilation matrix $E = M$ or D for the following reasons too:

(1) Many applications, such as image compression, employ wavelet bases in R^2. Although separable bases have a lot of advantages, they have a number of drawbacks. They are so special that they have very little design freedom, and separability imposes an unnecessary product structure on the plane, which is artificial for natural images. This preferred directions effect can create unpleasant artifacts that become obvious at high image compression ratios. Nonseparable wavelet bases offer the hope of a more isotropic analysis [1,3–7]. Any multiresolution analysis related to E corresponds to only one basic wavelet, and the basic wavelet has explicit expression. E-wavelets are nonseparable. Hence, one may hope for a more isotropic analysis than with the separable construction.

(2) In the bidimensional subband filtering schemes, perfect reconstruction does not depend on the dilation matrix E but only on the sublattice $\Gamma = EZ^2$ (different matrices may lead to the same Γ). There exist only two types of grids corresponding to all dilation matrices E with $|\det E| = 2$. The one is the quincunx sublattice generated by the integer combinations of $(1,1)^T$ and $(1,-1)^T$. The other is the column sublattice generated by the integer combinations of $(0,1)^T$ and $(2,0)^T$, which is of course equivalent to the row sublattice, by exchange of the coordinates. Both M and D correspond to the quincunx sublattice and column sublattice, respectively [1]. So, both M and D are representative to some extent.

(3) Even if for the same filter, the regularity of MRA wavelets is highly dependent on the choice of dilation matrix, and E-wavelets have the hope of being more regular than other A-wavelets with A satisfying $|\det A| = 2$ [1,3,5,8–10]. Especially, a construction of arbitrarily smooth orthogonal E-wavelets in R^2 was given in [5]. All this is also why we restrict ourselves to E-wavelet multipliers. In addition, the techniques used in this paper depend on the dilation matrix E. On one hand, for a general dilation matrix A with $|\det A| = 2$, maybe it is not very easy to find a suitable partition of R^2 like Lemma 2.5 below, which is important

throughout this paper. On the other hand, Examples 2.1 and 2.2 below play a crucial role in the proof of Theorem 1.1. Indeed, the union of the sets occurred in (3.3) and (3.4) in the proof of Theorem 1.1 is exactly R^2, which yields that the M-filter multiplier is unimodular. But, for any other $n \times n$ dilation matrix, it is unknown whether there exist examples like Examples 2.1 and 2.2 such that the union of the sets corresponding to (3.3) and (3.4) is exactly R^n. Therefore, it is unresolved whether a low pass filter multiplier related to any higher-dimensional dilation matrix is unimodular.

Let ψ_0 be an E-wavelet. $\mathcal{W}_{\psi_0}$ denotes the set of all E-wavelets ψ such that $|\hat{\psi}(\cdot)| = |\hat{\psi}(\cdot)|$. Define $\mathcal{M}_{\psi_0}$ by

$$\mathcal{M}_{\psi_0} = \bigl\{\psi\colon\ \hat{\psi}(\cdot) = v(\cdot)\hat{\psi}_0(\cdot),\ v \text{ is an } E\text{-wavelet multiplier}\bigr\}.$$

Suppose ψ is an MRA E-wavelet, and the MRA is generated by an E-scaling function ϕ. Then, by the same procedure as that of [11, Chapter 2], we have

$$\bigl|\hat{\phi}(\cdot)\bigr|^2 = \sum_{j=1}^{\infty}\bigl|\hat{\psi}\bigl((E^*)^j\cdot\bigr)\bigr|^2.$$

Therefore, although an MRA E-wavelet can correspond to several different scaling functions, the modules of their Fourier transforms are same.

Let ψ_0 be an MRA E-wavelet. We denote by $\mathcal{S}_{\psi_0}$ the set of all MRA E-wavelets ψ such that $|\hat{\phi}_0(\cdot)| = |\hat{\phi}(\cdot)|$, where ϕ_0 is an E-scaling function associated with the same MRA as ψ_0 and ϕ is an E-scaling function related with ψ in the same way. By the above arguments, $\mathcal{S}_{\psi_0}$ is independent of the choices of scaling function ϕ_0. Therefore, $\mathcal{S}_{\psi_0}$ is well defined.

Let us give an example. Suppose ψ_0 is an MRA E-wavelet, and ϕ_0 is a scaling function of the MRA satisfying $\hat{\phi}_0(\cdot) = \widetilde{H}_0((E^*)^{-1}\cdot)\hat{\phi}_0((E^*)^{-1}\cdot)$. Then, by the same procedure as that of [12], we have

$$\hat{\psi}_0(\cdot) = \rho_0\bigl((E^*)^{-1}\cdot\bigr)\widetilde{H}_1\bigl((E^*)^{-1}\cdot\bigr)\hat{\phi}_0\bigl((E^*)^{-1}\cdot\bigr),$$

where

$$\varepsilon_0 = \begin{cases} (1,0)^T & \text{if } E = M, \\ (0,1)^T & \text{if } E = D, \end{cases}$$

$$\widetilde{H}_1(\xi) = -e^{-i\xi_1}\,\overline{\widetilde{H}_0(\xi + \pi E^*\varepsilon_0)},$$

and ρ_0 is a $2\pi Z^2$-periodic function with $|\rho_0(\cdot)| = 1$. Define ϕ and ψ, respectively, by

$$\hat{\phi} = \bigl|\hat{\phi}_0\bigr|, \qquad \hat{\psi}(\cdot) = \rho_0\bigl((E^*)^{-1}\cdot\bigr)H_1\bigl((E^*)^{-1}\cdot\bigr)\hat{\phi}\bigl((E^*)^{-1}\cdot\bigr),$$

where $H_1(\xi) = -e^{-i\xi_1}|\widetilde{H}_0(\xi + \pi E^*\varepsilon_0)|$. Then, by the same procedure as that of [11, Theorem 5.2], $\psi \in \mathcal{S}_{\psi_0}$.

The main results of this paper can be stated as follows.

Theorem 1.1. *Let $E = M$ or D. Then the class of E-wavelet multipliers coincides with both the class of MRA E-wavelet multipliers and the class of E-scaling function multipliers. Moreover, a measurable function υ belongs to any one of these classes if and only if it is unimodular and $\upsilon(E^* \cdot)/\upsilon(\cdot)$ is $2\pi Z^2$-periodic. A measurable function μ is a low pass E-filter multiplier if and only if μ is unimodular and $2\pi Z^2$-periodic.*

Remark 1.1. Since these multipliers are unimodular, in all nontrivial cases the resulting wavelets and scaling functions will have infinite support, and if the υ is continuous, then the wavelets and scaling functions will have complex values.

Theorem 1.2. *Let $E = M$ or D, and ψ_0 be an MRA E-wavelet. Then $\mathcal{S}_{\psi_0} \subseteq \mathcal{M}_{\psi_0} \subseteq \mathcal{W}_{\psi_0}$.*

Theorem 1.3. *Let $E = M$ or D, and ψ_0 be an E-wavelet. Then $\mathcal{M}_{\psi_0}$ is arcwise connected; i.e., for any $\psi_1 \in \mathcal{M}_{\psi_0}$, there exists a continuous map $\theta : [0,1] \to L^2(R^2)$ such that $\theta(0) = \psi_0$, $\theta(1) = \psi_1$, and $\theta(t) \in \mathcal{M}_{\psi_0}$ for any $t \in [0,1]$.*

Remark 1.2. It is unresolved whether the relation $\subseteq$ can be replaced by the relation $=$ in Theorem 1.2.

2. Some auxiliary lemmas

We first give two examples of M-wavelets, which are frequently used in what follows.

Example 2.1. Define

$$\begin{aligned}
H_0^h(\xi) &= \frac{1 + e^{-i\xi_1}}{2}, \\
\hat{\phi}_h(\xi) &= \prod_{j=1}^{\infty} H_0^h(M^{-j}\xi), \\
\hat{\psi}_h(\xi) &= H_1^h(M^{-1}\xi)\hat{\phi}_h(M^{-1}\xi), \\
H_1^h(\xi) &= -e^{-i\xi_1}\overline{H_0^h\big(\xi + (\pi,\pi)^T\big)}
\end{aligned}$$

for $\xi = (\xi_1, \xi_2)^T \in R^2$. Then, it follows from [1] that ϕ_h generates an MRA related to M. H_0^h, ϕ_h, and ψ_h are called the low pass M-Haar filter, the M-Haar scaling function, and the M-Haar wavelet, respectively.

Example 2.2. Let H_0^s be a $2\pi Z^2$-periodic function such that

$$H_0^s(\xi) = \chi_{\{\xi:\ |\xi_1| \leqslant \pi/2,\ |\xi_2| \leqslant \pi\}}(\xi) \quad \text{for } \xi \in [-\pi, \pi]^2.$$

Define

$$\hat{\phi}_s(\xi) = \prod_{j=1}^{\infty} H_0^s(M^{-j}\xi),$$

$$\hat{\psi}_s(\xi) = H_1^s(M^{-1}\xi)\hat{\phi}_s(M^{-1}\xi),$$

$$H_1^s(\xi) = -e^{-i\xi_1}\overline{H_0^s\big(\xi + (\pi,\pi)^T\big)}$$

for $\xi = (\xi_1, \xi_2)^T \in R^2$. Then, it follows from [1] that ϕ_s generates an MRA related to M. H_0^s, ϕ_s, and ψ_s are called the low pass M-Shannon filter, the M-Shannon scaling function, and the M-Shannon wavelet, respectively.

By [13, Corollary 3.3], we have

Lemma 2.1. *ψ is an M-wavelet if and only if the following conditions hold*:

(1) $\|\psi\|_2 = 1$;
(2) $\sum_{j\in Z} |\hat{\psi}(M^j\cdot)|^2 = 1$;
(3) $\sum_{j=0}^{\infty} \hat{\psi}(M^j\cdot)\overline{\hat{\psi}(M^j(\cdot + 2\pi k))} = 0$ *for* $k \in Z^2 \setminus MZ^2$.

Analogously to the arguments used in [11, Theorems 3.2 and 5.2], we can obtain the following two propositions.

Lemma 2.2. *A function ψ is an MRA M-wavelet if and only if the following conditions hold*:

(1) $\|\psi\|_2 = 1$;
(2) $\sum_{j\in Z} |\hat{\psi}(M^j\cdot)|^2 = 1$;
(3) $\sum_{j=0}^{\infty} \hat{\psi}(M^j\cdot)\overline{\hat{\psi}(M^j(\cdot + 2\pi k))} = 0$ *for* $k \in Z^2 \setminus MZ^2$;
(4) $D_\psi(\cdot) = \sum_{j=1}^{\infty}\sum_{k\in Z^2} |\hat{\psi}(M^j(\cdot + 2\pi k))|^2 = 1$.

Lemma 2.3. *A function ϕ is an M-scaling function if and only if the following conditions hold*:

(1) $\sum_{k\in Z^2} |\hat{\phi}(\cdot + 2\pi k)|^2 = 1$;
(2) $\lim_{j\to\infty} |\hat{\phi}(M^{-j}\cdot)| = 1$;
(3) $\hat{\phi}(M\cdot) = H_0(\cdot)\hat{\phi}(\cdot)$ *for some $2\pi Z^2$-periodic function H_0.*

Lemma 2.4. *Suppose ϕ is a scaling function for an MRA $\{V_j\}_{j\in Z}$ related to M, and H_0 is the low pass M-filter. Then $\{\psi(\cdot - k)\colon k \in Z^2\}$ is an orthonormal basis for the orthogonal complementary subspace W_0 of V_0 in V_1 if and only if*

$$\hat{\psi}(\cdot) = \gamma(\cdot)H_1(M^{-1}\cdot)\hat{\phi}(M^{-1}\cdot)$$

for some $2\pi Z^2$-periodic, unimodular, and measurable function γ, where $H_1(\xi) = -e^{-i\xi_1}\overline{H_0(\xi + (\pi,\pi)^T)}$.

Proof. (Sufficiency) Define $\psi_0 = (H_1(M^{-1}\cdot)\hat{\phi}(M^{-1}\cdot))\check{}$. Then $\{\psi_0(\cdot - k)\colon k \in Z^2\}$ is an orthonormal basis for W_0. For $f \in L^2(R^2)$, it is easy to check that $\hat{f}(\cdot) = s(\cdot)\hat{\psi}_0(\cdot)$ for some $s \in L^2([-\pi,\pi]^2)$ if and only if $\hat{f}(\cdot) = \tilde{s}(\cdot)\psi(\cdot)$ for some $\tilde{s} \in L^2([-\pi,\pi]^2)$. Hence,

$$W_0 = \overline{\operatorname{span}\{\psi(\cdot - k)\colon k \in Z^2\}}.$$

Since $\{\psi_0(\cdot - k)\colon k \in Z^2\}$ is an orthonormal basis for W_0,

$$\sum_{k\in Z^2} \left|\hat{\psi}(\cdot + 2\pi k)\right|^2 = \sum_{k\in Z^2} \left|\hat{\psi}_0(\cdot + 2\pi k)\right|^2 = 1.$$

Therefore, $\{\psi(\cdot - k)\colon k \in Z^2\}$ is an orthonormal basis for W_0.

(Necessity) Suppose $\{\psi(\cdot - k)\colon k \in Z^2\}$ is an orthonormal basis for W_0. Since $\{\psi_0(\cdot - k)\colon k \in Z^2\}$ is also an orthonormal basis for W_0, there exists $\gamma \in L^2([-\pi,\pi]^2)$ such that $\hat{\psi}(\cdot) = \gamma(\cdot)\hat{\psi}_0(\cdot)$. Hence,

$$1 = \sum_{k\in Z^2} \left|\hat{\psi}(\cdot + 2\pi k)\right|^2 = |\gamma(\cdot)|^2 \sum_{k\in Z^2} \left|\hat{\psi}_0(\cdot + 2\pi k)\right|^2 = |\gamma(\cdot)|^2,$$

which finishes the proof. $\square$

Lemma 2.5. *Define $I_\pi = \{(\xi_1,\xi_2)^T\colon |\xi_1| \leqslant \pi,\ \pi \leqslant |\xi_1 + \xi_2| \leqslant 2\pi\}$. Then $\{M^j I_\pi\colon j \in Z\}$ is a partition of R^2.*

Proof. Define $S = I_\pi \cup M I_\pi$. It is easy to check that

$$\begin{aligned} S = {} & \left\{(\xi_1,\xi_2)^T\colon |\xi_1| \leqslant \pi,\ \pi \leqslant |\xi_1 + \xi_2| \leqslant 2\pi\right\} \\ & \cup \left\{(\xi_1,\xi_2)^T\colon \pi \leqslant |\xi_1| \leqslant 2\pi,\ |\xi_1 + \xi_2| \leqslant 2\pi\right\}, \end{aligned}$$

and, consequently,

$$2^j S \cap S = \emptyset \quad \text{for } 0 \neq j \in Z.$$

It is obvious that $I_\pi \cap M^{2j} I_\pi \subseteq S \cap 2^j S$ for $0 \neq j \in Z$, and that $I_\pi \cap M^{2j+1} I_\pi \subseteq S \cap 2^j M I_\pi \subseteq S \cap 2^j S$ for $j \in Z$.

Hence,

$$I_\pi \cap M^j I_\pi = \emptyset \quad \text{for } 0 \neq j \in Z. \tag{2.1}$$

For any $\xi \in R^2$ with $\xi_1 \neq 0$, there exists $j_1 \in Z$ such that $2^{j_1}\pi \leqslant |\xi_1| \leqslant 2^{j_1+1}\pi$. Since

$$M I_\pi = \left\{(\xi_1,\xi_2)^T\colon \pi \leqslant |\xi_1| \leqslant 2\pi,\ |\xi_1 + \xi_2| \leqslant 2\pi\right\}, \tag{2.2}$$

$\xi \in 2^{j_1} M I_\pi = M^{2j_1+1} I_\pi$ when $|\xi_1 + \xi_2| \leqslant 2^{j_1+1}\pi$. When $|\xi_1 + \xi_2| > 2^{j_1+1}\pi$, there exists j_2, $j_1 + 1 \leqslant j_2 \in Z$ such that $2^{j_2}\pi \leqslant |\xi_1 + \xi_2| \leqslant 2^{j_2+1}\pi$. Note that since $|\xi_1| \leqslant 2^{j_2}\pi$, we obtain that $\xi \in 2^{j_2} I_\pi = M^{2j_2} I_\pi$. Therefore,

$$R^2 = \bigcup_{j \in Z} M^j I_\pi. \tag{2.3}$$

This together with (2.1) yields that $\{M^j I_\pi\colon\ j \in Z\}$ is a partition of R^2. The proof is completed. □

Lemma 2.6. *Assume that v is a unimodular and measurable function on R^2. Then there exists a unimodular and measurable function t on R^2 satisfying*

$$v(\cdot) = t(M\cdot)\overline{t(\cdot)}.$$

Proof. Let I_π be defined as in Lemma 2.5, and t be any unimodular measurable function defined on I_π. Define

$$t(\cdot) = t(M^{-1}\cdot)v(M^{-1}\cdot) \quad \text{on } M^j I_\pi,\ j \geqslant 1,$$

and

$$t(\cdot) = t(M\cdot)\overline{v(\cdot)} \quad \text{on } M^j I_\pi,\ j \leqslant -1.$$

Then, by Lemma 2.5, t is well defined on R^2. By the definition of t, t is unimodular, and $v(\cdot) = t(M\cdot)\overline{t(\cdot)}$. The proof is completed. □

Lemma 2.7. *Assume that v is an M-wavelet multiplier, or an MRA M-wavelet multiplier, or an M-scaling function multiplier. Then v is unimodular.*

Proof. First, we show that $|v(\cdot)| \leqslant 1$. Since

$$\left\{\xi \in R^2\colon\ |v(\xi)| > 1\right\} = \bigcup_{n=1}^{\infty} F_n, \qquad F_n = \left\{\xi \in R^2\colon\ |v(\xi)| \geqslant 1 + \frac{1}{n}\right\},$$

it suffices to show that $|F_n| = 0$ for $n = 1, 2, 3, \ldots$.

Let ψ be an M-wavelet such that $\hat{\psi}(\cdot) \neq 0$ a.e. on R^2 (we can choose ψ to be the M-Haar wavelet). Then, for any fixed $n \in \mathbf{N}$, there exists an $\varepsilon > 0$ such that

$$\left|\left\{\xi \in R^2\colon\ |v(\xi)| > \varepsilon\right\} \cap F_n\right| \geqslant \frac{1}{2}|F_n|. \tag{2.4}$$

Take $N \in \mathbf{N}$ such that $\varepsilon(1 + 1/n)^N > 1$; then

$$\left|(v(\xi))^N \hat{\psi}(\xi)\right| > 1$$

for $\xi \in \{\xi \in R^2\colon\ |\hat{\psi}(\xi)| > \varepsilon\} \cap F_n$.

Since $(v(\cdot))^N \hat{\psi}(\cdot)$ is the Fourier transform of an M-wavelet, $|(v(\cdot))^N \hat{\psi}(\cdot)| \leqslant 1$ a.e. on R^2. Hence, $|\{\xi \in R^2\colon\ |\hat{\psi}(\xi)| > \varepsilon\} \cap F_n| = 0$. This together with (2.4) yields that $|F_n| = 0$. Therefore, $|v(\cdot)| \leqslant 1$.

Secondly, $|v(\cdot)| = 1$. Since ψ and $(v\hat{\psi})\check{}$ are M-wavelets,

$$\int_{R^2} d\xi\, \bigl(1 - |v(\xi)|^2\bigr)\bigl|\hat{\psi}(\xi)\bigr|^2 = 0,$$

which implies that $|v(\cdot)| = 1$ due to $|v(\cdot)| \leqslant 1$.

It is easy to see that this argument also shows that an MRA M-wavelet multiplier must be unimodular. If v is an M-scaling function multiplier, using the M-Haar scaling function instead of the M-Haar wavelet we can show that v is unimodular analogously. The proof is completed. □

Lemma 2.8. *Assume that ϕ is an M-scaling function with H being its filter. Define*

$$F = \bigl\{\xi \in R^2\colon \phi(M^{l+1}\xi) = H(M^l\xi)\phi(M^l\xi) \text{ for } l \in Z\bigr\},$$
$$E = \bigl\{\xi \in F\colon \phi(\xi) \neq 0\bigr\},$$
$$\Delta_0 = E, \qquad \Delta_n = M^n E \setminus M^{n-1}E \quad \text{for } n \geqslant 1.$$

Then $\{\Delta_n\colon n \geqslant 0\}$ is a partition of R^2.

Proof. Let

$$F_l = \bigl\{\xi \in R^2\colon \phi(M^{l+1}\xi) = H(M^l\xi)\phi(M^l\xi)\bigr\}$$

for each $l \in Z$. Then $F_l = M^{-l}F_0$, $|R^2 \setminus F_0| = 0$, and, consequently,

$$|R^2 \setminus F| = \Bigl|\bigcup_{l\in Z} R^2 \setminus F_l\Bigr| = \Bigl|\bigcup_{l\in Z} M^{-l}(R^2 \setminus F_0)\Bigr| = 0. \tag{2.5}$$

By the definition of E, it is easy to check that

$$M^n E \subseteq M^{n+1}E \quad \text{for } n = 0, 1, 2, \ldots. \tag{2.6}$$

Let $K = F \setminus (\bigcup_{n=0}^{\infty} M^n E)$. If $\xi \in K$, then $\xi \in F$, and thus $M^n\xi \in F$ for $n \in Z$. But $M^{-n}\xi \notin E$. So, $\hat{\phi}(M^{-n}\xi) = 0$ for $n \geqslant 0$. Hence,

$$\chi_K(\cdot)\hat{\phi}(M^{-n}\cdot) = 0 \quad \text{for } n \geqslant 0, \tag{2.7}$$

which implies that $|K| = 0$ since the Fourier transform of $2^{n/2}\phi(M^n \cdot - k)$ is

$$2^{-n/2}e^{-ikM^{-n}\cdot}\phi(M^{-n}\cdot)$$

for $n \in Z$ and $k \in Z^2$.

This together with (2.5) and (2.6) finishes the proof of the lemma. □

Lemma 2.9. *Let $\psi \in L^2(R^2)$, M, D be defined as in* (1.1) *and* (1.2), *respectively, and*

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Define $\tilde{\psi}(\cdot) = \psi(P\cdot)$. Then ψ is an M-wavelet if and only if $\tilde{\psi}$ is a D-wavelet.

Proof. Note that $PDP^{-1} = M$; it is easy to check that

$$\int_{R^2} dx\, 2^{j/2}\tilde{\psi}(D^j x - k)\overline{2^{j'/2}\tilde{\psi}(D^{j'} x - k')}$$
$$= \int_{R^2} dx\, 2^{j/2}\psi(M^j x - Pk)\overline{2^{j'/2}\psi(M^{j'} x - Pk')} \tag{2.8}$$

for $j, j' \in Z$, $k, k' \in Z^2$.

Suppose ψ is an M-wavelet. Then, by (2.8), $\tilde{\psi}$ generates an M-orthonormal system. For any $f \in L^2(R^2)$, $f(P^{-1}\cdot) \in L^2(R^2)$. Hence,

$$f(P^{-1}\cdot) = \sum_{j\in Z,\ k\in Z^2} f_{j,k} 2^{j/2}\psi(M^j \cdot - k)$$
$$= \sum_{j\in Z,\ k\in Z^2} f_{j,k} 2^{j/2}\tilde{\psi}(P^{-1}M^j \cdot - P^{-1}k)$$

for some $\{f_{j,k}\} \in l^2(Z \times Z^2)$, and, consequently,

$$f(\cdot) = \sum_{j\in Z,\ k\in Z^2} f_{j,k} 2^{j/2}\tilde{\psi}(P^{-1}M^j P \cdot - P^{-1}k)$$
$$= \sum_{j\in Z,\ k\in Z^2} f_{j,k} 2^{j/2}\tilde{\psi}(D^j \cdot - P^{-1}k).$$

Therefore, $\tilde{\psi}$ is a D-wavelet. The necessity is showed.

The sufficiency can be showed analogously. □

Lemma 2.10. *Let*

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Then $\phi \in L^2(R^2)$ is an M-scaling function if and only if $\tilde{\phi}(\cdot) = \phi(P\cdot)$ is a D-scaling function, and H_0 is a low pass M-filter if and only if $\widetilde{H}_0(\cdot) = H_0(Q\cdot)$ is a low pass D-filter.

The proof is omitted.

Lemma 2.11. *Let*

$$P_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then a measurable function v is a D-wavelet multiplier (an MRA D-wavelet multiplier, or a D-scaling function multiplier) if and only if $v_1(\cdot) = v(P_1\cdot)$ is an M-wavelet multiplier (an MRA M-wavelet multiplier, or an M-scaling function *multiplier), and a measurable function μ is a low pass D-filter multiplier if and only if $\mu_1(\cdot) = \mu(P_1\cdot)$ is a low pass M-filter multiplier.*

Proof. The characterization of scaling function multiplier can be given analogously to that of wavelet multiplier. Moreover, by Lemmas 2.2 and 2.7, the characterization of MRA wavelet multiplier holds if we give the characterization of wavelet multiplier. Hence, it is enough to give the characterizations of wavelet multiplier and low pass filter multiplier.

First, we give the characterization of wavelet multiplier. Suppose v is a D-wavelet multiplier. Let ψ be an M-wavelet; then, by Lemma 2.9, $\hat{\psi}(P_1^{-1}\cdot)$ is the Fourier transform of a D-wavelet. Hence, $v(\cdot)\hat{\psi}(P_1^{-1}\cdot)$ is also the Fourier transform of a D-wavelet, which implies that $v_1(\cdot)\hat{\psi}(\cdot)$ is the Fourier transform of an M-wavelet. Therefore v_1 is an M-wavelet multiplier.

Conversely, if v_1 is an M-wavelet multiplier, then it is a D-wavelet multiplier by the similar arguments to the above.

Secondly, we give the characterization of filter multiplier. Suppose μ is a low pass D-filter multiplier. Let H_0 be a low pass M-filter; then $H_0(Q\cdot)$ is a low pass D-filter, and, consequently, $\mu(\cdot)H_0(Q\cdot)$ is also a low pass D-filter, where Q is defined as in Lemma 2.10. Again, by Lemma 2.10, $\mu_1(\cdot)H_0(\cdot)$ is a low pass M-filter. Hence, μ_1 is a low pass M-filter multiplier.

Analogously, μ is a low pass D-filter multiplier if μ_1 is a low pass M-filter multiplier. The proof is completed. $\square$

3. Proofs of theorems

Proof of Theorem 1.1. By Lemma 2.11, it suffices to show the theorem holds for $E = M$. We begin with the characterization of the M-filter multipliers.

First, suppose μ is unimodular and $2\pi Z^2$-periodic. It follows from Lemma 2.6 that

$$\mu(\cdot) = t(M\cdot)\overline{t(\cdot)} \tag{3.1}$$

for some unimodular measurable function t on R^2.

Let H_0 be a low pass M-filter defined by the scaling function ϕ. Define $\tilde{\phi}$ by $\hat{\tilde{\phi}}(\cdot) = t(\cdot)\hat{\phi}(\cdot)$. Then it follows from (3.1) that

$$\hat{\tilde{\phi}}(M\cdot) = \mu(\cdot)H_0(\cdot)\hat{\tilde{\phi}}(\cdot). \tag{3.2}$$

This, by Lemma 2.3, implies that $\tilde{\phi}$ is an M-scaling function, and that μH_0 is a low pass M-filter. Therefore, μ is an M-filter multiplier.

Secondly, suppose μ is an M-filter multiplier. Let H_0^s be the low pass M-Shannon filter. Then μH_0^s is a low pass M-filter, and, consequently,

$$\left|\mu(\cdot)H_0^s(\cdot)\right|^2 + \left|\mu\left(\cdot + (\pi,\pi)^T\right)H_0^s\left(\cdot + (\pi,\pi)^T\right)\right|^2 = 1.$$

Observing that

$$H_0^s(\cdot) = 1, \qquad H_0^s\big(\cdot + (\pi,\pi)^T\big) = 0$$

on

$$\left\{(\xi_1,\xi_2) \in R^2\colon -\frac{\pi}{2} + 2\pi k \leqslant \xi_1 \leqslant \frac{\pi}{2} + 2\pi k,\ k \in Z\right\},$$

we obtain that

$$|\mu(\cdot)| = 1 \tag{3.3}$$

on

$$\left\{(\xi_1,\xi_2) \in R^2\colon -\frac{\pi}{2} + 2\pi k \leqslant \xi_1 \leqslant \frac{\pi}{2} + 2\pi k,\ k \in Z\right\}.$$

Analogously, using the low pass M-Haar filter H_0^h instead of H_0^s, we obtain that

$$|\mu(\cdot)| = 1 \tag{3.4}$$

on

$$\left\{(\xi_1,\xi_2) \in R^2\colon \frac{\pi}{2} + 2\pi k \leqslant \xi_1 \leqslant \frac{3\pi}{2} + 2\pi k,\ k \in Z\right\},$$

which together with (3.3) leads to

$$|\mu(\cdot)| = 1.$$

Since μH_0^h is a low pass M-filter and $H_0^h(\cdot) \neq 0$, we obtain that $\mu(\cdot) = \mu(\cdot) \times H_0^h(\cdot)/H_0^h(\cdot)$, which is $2\pi Z^2$-periodic. Therefore, μ is unimodular and $2\pi Z^2$-periodic.

We now turn to the characterization of the other multipliers. Suppose v is a unimodular function such that $v(M\cdot)/v(\cdot)$ is $2\pi Z^2$-periodic. To show that v is an M-wavelet multiplier, it suffices to show that $\hat{\tilde{\psi}}(\cdot) = v(\cdot)\hat{\psi}(\cdot)$ is the Fourier transform of an M-wavelet whenever ψ is an M-wavelet. Obviously, $\tilde{\psi}$ satisfies (1) and (2) of Lemma 2.1. Hence, it suffices to show that $\tilde{\psi}$ satisfies (3) of Lemma 2.1.

$$\begin{aligned}
&v(M^j\cdot)\overline{v\big(M^j(\cdot + 2\pi k)\big)} \\
&\quad = v(M^j\cdot)\overline{v(M^{j-1})}v\big(M^{j-1}(\cdot + 2\pi k)\big)\overline{v\big(M^j(\cdot + 2\pi k)\big)} \\
&\qquad \times v(M^{j-1})\overline{v\big(M^{j-1}(\cdot + 2\pi k)\big)} \\
&\quad = v(M^{j-1})\overline{v(M^{j-1}(\cdot + 2\pi k)\big)} \\
&\qquad \vdots \\
&\quad = v(\cdot)\overline{v(\cdot + 2\pi k)}
\end{aligned}$$

for $j \geqslant 1$, $k \in Z^2$. This together with Lemma 2.1 leads to that

$$\sum_{j=0}^{\infty} \hat{\tilde{\psi}}(M^j \cdot)\overline{\hat{\tilde{\psi}}\big(M^j(\cdot + 2\pi k)\big)}$$
$$= v(\cdot)\overline{v(\cdot + 2\pi k)} \sum_{j=0}^{\infty} \hat{\psi}(M^j \cdot)\overline{\hat{\psi}\big(M^j(\cdot + 2\pi k)\big)} = 0$$

for $k \in Z^2 \setminus MZ^2$. Therefore, v is an M-wavelet multiplier.

It follows from Lemmas 2.2 and 2.7 that an M-wavelet multiplier must be an MRA M-wavelet multiplier.

In the next, we show that if v is an MRA M-wavelet multiplier, then v is unimodular and $v(M\cdot)/v(\cdot)$ is $2\pi Z^2$-periodic. By Lemma 2.7, it suffices to show that $v(M\cdot)/v(\cdot)$ is $2\pi Z^2$-periodic.

Let ψ_h, ϕ_h, H_0^h, and H_0^h be defined as in Example 2.1. Define $\tilde{\psi}$ by

$$\hat{\tilde{\psi}}(\xi) = -e^{-i(\xi_1+\xi_2)/2} v(\xi)\big|\hat{\psi}_h(\xi)\big|;$$

then $\tilde{\psi}$ is an MRA M-wavelet. By Lemma 2.3, we can find an M-scaling function $\tilde{\phi}$ with the corresponding filter $\widetilde{H}_0$ such that

$$-e^{-i(\xi_1+\xi_2)/2} v(\xi)\big|\hat{\psi}_h(\xi)\big| = \widetilde{H}_1(M^{-1}\xi)\hat{\tilde{\psi}}(M^{-1}\xi), \tag{3.5}$$

where $\widetilde{H}_1(\xi) = -e^{-i\xi_1}\overline{\widetilde{H}_0(\xi + (\pi,\pi)^T)}\tilde{\gamma}(M\xi)$ for some unimodular and $2\pi Z^2$-periodic function $\tilde{\gamma}$.

Since v is unimodular, $|\hat{\tilde{\psi}}(\cdot)| = |\hat{\psi}_h|$, and, consequently,

$$\big|\hat{\tilde{\phi}}(\cdot)\big|^2 = \sum_{j=1}^{\infty}\big|\hat{\tilde{\psi}}_h(M^j \cdot)\big|^2 = \sum_{j=1}^{\infty}\big|\hat{\tilde{\psi}}_h(M^j \cdot)\big|^2 = |\hat{\phi}_h|^2,$$
$$\big|\widetilde{H}_0(\cdot)\big| = \big|H_0^h(\cdot)\big|.$$

This together with (3.5) leads to

$$\frac{v(M\cdot)}{v(\cdot)} = \frac{\tilde{\gamma}(M\cdot)\overline{\widetilde{H}_0(\cdot + (\pi,\pi)^T)}\hat{\tilde{\phi}}(\cdot)|\hat{\psi}_h|(\cdot)|}{\tilde{\gamma}(\cdot)\overline{\widetilde{H}_0(M^{-1}\cdot + (\pi,\pi)^T)}\hat{\tilde{\phi}}(M^{-1}\cdot)|\hat{\psi}_h(M\cdot)|}$$
$$= \frac{\tilde{\gamma}(M\cdot)\overline{\widetilde{H}_0(\cdot + (\pi,\pi)^T)}}{\tilde{\gamma}(\cdot)|\widetilde{H}_0(\cdot + (\pi,\pi)^T)|} \frac{\widetilde{H}_0(M^{-1}\cdot)|\widetilde{H}_0(M^{-1}\cdot + (\pi,\pi)^T)|}{|\widetilde{H}_0(M^{-1}\cdot)|\overline{\widetilde{H}_0(M^{-1}\cdot + (\pi,\pi)^T)}}.$$

Note that

$$\frac{\widetilde{H}_0(M^{-1}\cdot)|\widetilde{H}_0(M^{-1}\cdot + (\pi,\pi)^T)|}{|\widetilde{H}_0(M^{-1}\cdot)|\overline{\widetilde{H}_0(M^{-1}\cdot + (\pi,\pi)^T)}} = \frac{\widetilde{H}_0(M^{-1}\cdot + (\pi,\pi)^T))|\widetilde{H}_0(M^{-1}\cdot)}{|\widetilde{H}_0(M^{-1}\cdot + (\pi,\pi)^T)|\overline{\widetilde{H}_0(M^{-1}\cdot)}}.$$

It is easy to check that

$$\frac{\widetilde{H}_0(M^{-1}\cdot)|\widetilde{H}_0(M^{-1}\cdot+(\pi,\pi)^T)|}{|\widetilde{H}_0(M^{-1}\cdot)|\overline{\widetilde{H}_0(M^{-1}\cdot+(\pi,\pi)^T)}}$$

is $2\pi Z^2$-periodic, and, consequently, $v(M\cdot)/v(\cdot)$ is $2\pi Z^2$-periodic. Therefore, v is unimodular and $v(M\cdot)/v(\cdot)$ is $2\pi Z^2$-periodic.

We now turn to the characterization of the M-scaling function multipliers. First, we show that v must be an M-scaling function multiplier if v is unimodular and $v(M\cdot)/v(\cdot)$ is $2\pi Z^2$-periodic.

Suppose ϕ is an M-scaling function with the corresponding filter H_0. Define $\tilde{\phi}$ by $\hat{\tilde{\phi}}(\cdot)=v(\cdot)\hat{\phi}(\cdot)$. Then

$$\hat{\tilde{\phi}}(M\cdot)=\frac{H_0(\cdot)v(M\cdot)\hat{\tilde{\phi}}(\cdot)}{v(\cdot)},$$

and $H_0(\cdot)v(M\cdot)/v(\cdot)$ is $2\pi Z^2$-periodic. Applying Lemma 2.3 to $\tilde{\phi}$, we obtain that $\tilde{\phi}$ is an M-scaling function. Therefore, v is an M-scaling function multiplier.

Secondly, we show that v is unimodular and $v(M\cdot)/v(\cdot)$ is $2\pi Z^2$-periodic if v is an M-scaling function multiplier.

By Lemma 2.7, it suffices to show that $v(M\cdot)/v(\cdot)$ is $2\pi Z^2$-periodic. Let ϕ_h and H_0^h be defined as in Example 2.1. Define $\tilde{\phi}$ by $\hat{\tilde{\phi}}(\cdot)=v(\cdot)\hat{\phi}_h(\cdot)$. Then $\tilde{\phi}$ is an M-scaling function. Suppose $\hat{\tilde{\phi}}(M\cdot)=\widetilde{H}_0(\cdot)\hat{\tilde{\phi}}(\cdot)$; then it is easy to check that

$$\widetilde{H}_0(\cdot)v(\cdot)\hat{\phi}_h(\cdot)=v(M\cdot)H_0^h(\cdot)\hat{\phi}_h(\cdot).$$

Hence,

$$\frac{v(M\cdot)}{v(\cdot)}=\frac{\widetilde{H}_0(\cdot)}{H_0^h(\cdot)},$$

which is $2\pi Z^2$-periodic. The proof is completed. □

Proof of Theorem 1.2. By Lemmas 2.9–2.11, it suffices to show the theorem holds for $E=M$. It is obvious that $\mathcal{M}_{\psi_0}\subseteq\mathcal{W}_{\psi_0}$. In the next, we show that $\mathcal{S}_{\psi_0}\subseteq\mathcal{M}_{\psi_0}$. Suppose $\psi_1\in\mathcal{S}_{\psi_0}$, and

$$\hat{\psi}_j(\xi)=-\gamma(\xi)e^{-i(\xi_1+\xi_2)/2}\overline{H_j\big(M^{-1}\xi+(\pi,\pi)^T\big)}\hat{\phi}_j(M^{-1}\xi)$$

for some unimodular $2\pi Z^2$-periodic function γ and $j=0,1$. Then $|\hat{\phi}_0|=|\hat{\phi}_1|$.

Define $\tilde{\phi}$ by $\hat{\tilde{\phi}}(\cdot)=|\hat{\phi}_0(\cdot)|$. Then, by Lemma 2.3, $\tilde{\phi}$ is an M-scaling function with the corresponding filter $\widetilde{H}=|H_0|$. Define $\tilde{\psi}$ by

$$\hat{\tilde{\psi}}(\xi)=-\gamma(\xi)e^{-i(\xi_1+\xi_2)/2}\overline{\widetilde{H}\big(M^{-1}\xi+(\pi,\pi)^T\big)}\hat{\tilde{\phi}}(M^{-1}\xi).$$

In order to show that $\psi_1 \in \mathcal{M}_{\psi_0}$, it suffices to show that there exist wavelet multipliers v_1 and v_2 such that

$$v_j \hat{\tilde{\psi}} = \hat{\psi} \quad \text{for } j = 0, 1.$$

In fact, at this time, $\hat{\psi}_1 = v_1 \bar{v}_0 \hat{\psi}_0$, $v_1 \bar{v}_0$ is an M-wavelet multiplier. This implies that $\psi_1 \in \mathcal{M}_{\psi_0}$.

We shall divide two cases to construct the M-wavelet multiplier v_1 such that

$$v_1 \hat{\tilde{\psi}} = \hat{\psi}_1. \tag{3.6}$$

v_0 can be constructed in the same way.

Case 1. $\{\xi \in R^2\colon\ \hat{\phi}_0(\xi) \neq 0\} = R^2$. In this case, $t(\cdot) = \hat{\tilde{\phi}}(\cdot)/\hat{\phi}_1(\cdot)$ is well defined and unimodular, and $\widetilde{H}(\cdot) = |H_1(\cdot)| = |H_0(\cdot)| \neq 0$.

Putting

$$v_1(\cdot) = \overline{\left(\frac{H_1(M^{-1} \cdot + (\pi,\pi)^T)t(M^{-1}\cdot)}{\widetilde{H}(M^{-1} \cdot + (\pi,\pi)^T)}\right)},$$

then v_1 is unimodular and

$$\frac{v_1(M\cdot)}{v_1(\cdot)} = \overline{\left(\frac{H_1(\cdot + (\pi,\pi)^T)\widetilde{H}(M^{-1}\cdot)\widetilde{H}(M^{-1} \cdot + (\pi,\pi)^T)}{\widetilde{H}(\cdot + (\pi,\pi)^T)H_1(M^{-1}\cdot)H_1(M^{-1} \cdot + (\pi,\pi)^T)}\right)},$$

which is $2\pi Z^2$-periodic. Hence, v_1 is an M-wavelet multiplier. It is easy to check that $v_1 \hat{\tilde{\psi}} = \hat{\psi}_1$. (3.6) follows.

Case 2. $\{\xi \in R^2\colon\ \hat{\phi}_0(\xi) \neq 0\} \neq R^2$. Define the $2\pi Z^2$-periodic function μ such that, for $\xi \in [-\pi,\pi]^2$, $\mu(\xi) = H_1(\xi)/\widetilde{H}(\xi)$ when $H_1(\xi) \neq 0$, $\mu(\xi) = 1$ when $H_1(\xi) = 0$.

In order to find a v_1 satisfying (3.6), it suffices to construct a measurable function t such that

$$|t(\cdot)| = 1, \tag{3.7}$$

$$\hat{\tilde{\phi}}(\cdot) = t(\cdot)\hat{\phi}_1(\cdot), \tag{3.8}$$

$$\overline{\mu(\cdot)} = t(M\cdot)\overline{t(\cdot)}. \tag{3.9}$$

Indeed, if (3.7)–(3.9) hold, then, taking $v_1(\cdot) = \overline{\mu(M^{-1} \cdot + (\pi,\pi)^T)t(M^{-1}\cdot)}$, we obtain that $v_1 \hat{\tilde{\psi}} = \hat{\psi}$ and

$$\frac{v_1(M\cdot)}{v_1(\cdot)} = \overline{\mu\big(\cdot + (\pi,\pi)^T\big)}\mu(M^{-1}\cdot)\mu\big(M^{-1} \cdot + (\pi,\pi)^T\big),$$

which is $2\pi Z^2$-periodic. It is obvious that v_1 is unimodular. Hence v_1 satisfies (3.6).

In the next, we shall construct t satisfying (3.7)–(3.9).

Using ϕ instead of ϕ_1 in Lemma 2.8, we obtain the partition $\{\Delta_n\colon\ n \geqslant 0\}$ of R^2. Define the function t by

$$t(\xi) = \frac{\hat{\tilde{\phi}}(\xi)}{\hat{\phi}(\xi)} \quad \text{for } \xi \in \Delta_0,$$
$$t(\xi) = \overline{\mu(M^{-1}\xi)}\, t(M^{-1}\xi) \quad \text{for } \xi \in \Delta_n,\ n \geqslant 1.$$

Then t satisfies (3.7) and (3.9). We claim that t satisfies (3.8), which can be showed by induction. It is obvious that (3.8) holds for $\xi \in \Delta_0$. Assuming (3.8) holds for $\xi \in \Delta_n$, then, for $\xi \in \Delta_{n+1}$,

$$\begin{aligned}\tilde{\phi}(\xi) &= \widetilde{H}(M^{-1}\xi)\hat{\tilde{\phi}}(M^{-1}\xi) = \widetilde{H}(M^{-1}\xi)t(M^{-1}\xi)\hat{\phi}_1(M^{-1}\xi)\\ &= H_1(M^{-1}\xi)\overline{\mu(M^{-1}\xi)}\,t(M^{-1}\xi)\hat{\phi}_1(M^{-1}\xi) = t(\xi)\hat{\phi}_1(\xi).\end{aligned}$$

By Lemma 2.8, (3.8) follows. The proof is completed. $\square$

Proof of Theorem 1.3. By Lemmas 2.9–2.11, it suffices to show that the theorem holds for $E = M$. Suppose $\psi_1 \in \mathcal{M}_{\psi_0}$; then $\hat{\psi}_1 = v\hat{\psi}_0$ for some wavelet multiplier v. Define the function λ such that $0 \leqslant \lambda(\xi) < 2\pi$ and $v(\xi) = e^{i\lambda(\xi)}$ for $\xi \in I_\pi$, where I_π is defined as in Lemma 2.5. Since v is an M-wavelet multiplier, $v(M\cdot)/v(\cdot) = e^{i\beta(\cdot)}$ for some $2\pi Z^2$-periodic real function β on R^2. By Lemma 2.3, $\{M^j I_\pi\colon\ j \in Z\}$ is a partition of R^2. Hence, we can extend λ to R^2 in the following way:

$$\lambda(\xi) = \lambda(M^{-1}\xi) + \beta(M^{-1}\xi) \quad \text{for } \xi \in M^{j+1}I_\pi,\ j \geqslant 0,$$
$$\lambda(\xi) = \lambda(M\xi) - \beta(\xi) \quad \text{for } \xi \in M^j I_\pi,\ j < 0.$$

It is easy to check that $v(\cdot) = e^{i\lambda(\cdot)}$.

Define the map $\theta : [0,1] \to L^2(R^2)$ by

$$\theta(t) = \left(v_t\hat{\psi}_0\right)^{\vee} \quad \text{for } t \in [0,1],$$

where $v_t(\cdot) = e^{it\lambda(\cdot)}$. Then,

$$\theta(0) = \psi_0, \qquad \theta(1) = \psi_1. \tag{3.10}$$

It is obvious that $v(M\cdot)/v(\cdot) = e^{it\beta(\cdot)}$, which is $2\pi Z^2$-periodic. Hence, v_t is an M-wavelet multiplier, and, consequently,

$$\theta(t) \in \mathcal{M}_{\psi_0} \quad \text{for } 0 \leqslant t \leqslant 1. \tag{3.11}$$

Since $|\widehat{\theta(t)}(\cdot) - \widehat{\theta(s)}(\cdot)| \leqslant 4|\hat{\psi}_0(\cdot)|^2$ for $0 \leqslant t, s \leqslant 1$, by the Lebesgue dominated convergence theorem and Plancheral's theorem,

$$\lim_{t\to s}\left\|\theta(t) - \theta(s)\right\|_2 = 0.$$

Hence, θ is a continuous map. This together with (3.10) and (3.11) implies $\mathcal{M}_{\psi_0}$ is arcwise connected. The proof is completed. $\square$

Example 3.1. For any $2\pi Z^2$-periodic function ω on R^2, define μ by

$$\mu(\xi)=\begin{cases}\frac{\omega(\xi)}{|\omega(\xi)|} & \text{if } \omega(\xi)\neq 0,\\ 1 & \text{if } \omega(\xi)=0.\end{cases}$$

Then, by Theorem 1.1, μ is both an E-wavelet multiplier and a low pass E-filter multiplier.

Example 3.2. Define v on R^2 by

$$v(\xi)=\begin{cases}c_1 & \text{if } \xi\in\bigcup_{j\in Z}(E^*)^j([0,\pi]^2),\\ c_2 & \text{if } \xi\notin\bigcup_{j\in Z}(E^*)^j([0,\pi]^2),\end{cases}$$

where c_1 and c_2 are two constants satisfying conditions $|c_1|=|c_2|=1$ and $c_1\neq c_2$. Then, it is easy to check that $\xi\in\bigcup_{j\in Z}(E^*)^j([0,\pi]^2)$ if and only if

$$E^*\xi\in\bigcup_{j\in Z}(E^*)^j\big([0,\pi]^2\big),$$

and

$$\xi-(2\pi,0)^T\notin\bigcup_{j\in Z}(E^*)^j\big([0,\pi]^2\big)$$

for $\xi\in[0,\pi]^2\subset\bigcup_{j\in Z}(E^*)^j([0,\pi]^2)$. Therefore, $v(E^*\cdot)/v(\cdot)=1$, but v is not $2\pi Z^2$-periodic. By Theorem 1.1, v is an E-wavelet multiplier, an MRA E-wavelet multiplier, and also an E-scaling function multiplier.

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